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ON GENERALIZED SPACE DIFFERENTIATION OF THE SECOND ORDER.

By B. O. Peirce.

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If one has to investigate the strength of a field of force defined by a given scalar potential function, or to study the flow of electricity in a massive conductor under given conditions, or to apply Green's Theorem to given functions in the space bounded by a given closed surface, or, indeed, to treat any one of a large number of problems in Mathematical Physics or in Analysis, one often needs to find the numerical value at a point, of the derivative of a point function taken in a given direction. This has given rise to the familiar idea of simple space differentiation and of the normal derivative of one scalar function with respect to another; indeed the properties of the first and of the higher space derivatives of a function of n variables taken with respect to any *fixed* direction in n dimensional space, have been treated very clearly and exhaustively by Czuber.*

It is sometimes desirable to use also the conception of general space derivatives of the second order. This is the case, for instance, when one is determining the rate of change of the intensity of a conservative field of force at a point which is moving, either along a curved line of force or on a curved surface related to such lines in a prescribed manner. It is easy to define the general space derivative of any order of a given function.

This paper discusses very briefly a few elementary facts with regard to generalized space differentiation of the second order, and treats first, for the sake of simplicity, differentiation of functions of two variables, in the plane of those variables.

Plane Differentiation.

Let there be in the xy plane two independent families of curves $(u = c, v = k)$ such that in the domain, R, one and only one curve of

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* E. Czuber, Wienerberichte, p. 1417 (1892).

each family passes through every point and no curve of either family has anywhere a multiple point. At every point, P, in the domain, the two curves (one of the u family and one of the v family) which pass through the point indicate two directions, s_1, s_2, and if the sense of each of these be determined by any convenient convention, they may be defined by pairs of direction cosines (l_1, m_1), (l_2, m_2), where l_1, m_1, l_2, m_2 are given scalar functions such that at every point

$$l_1^2 + m_1^2 = 1, \quad l_2^2 + m_2^2 = 1. \tag{1}$$

If Ω is any scalar function of the coördinates which within R has finite derivatives of the first and second orders with respect to these coördinates, the derivative of Ω at P in the direction s_1 is the value at the point of the quantity

$$l_1 \cdot \frac{\partial \Omega}{\partial x} + m_1 \cdot \frac{\partial \Omega}{\partial y} \tag{2}$$

and this new scalar function of x and y may be conveniently indicated by the expression $[D_{s_1}\Omega]_P$. If P' is a point on the u curve which passes through P, taken near P and in the sense of the direction s_1, $[D_{s_1}\Omega]_P$ is the limit, as P' approaches P, of $\dfrac{\Omega_{P'} - \Omega_P}{PP'}$.

If on the curve of the second (or v) family which passes through P, a point Q be taken near P and in the sense of the direction s_2, the limit, as Q approaches P, of the quantity

$$\frac{[D_{s_1}\Omega]_Q - [D_{s_1}\Omega]_P}{PQ} \tag{3}$$

may be indicated by the expression $[D_{s_2} D_{s_1}\Omega]_P$, and this is the second derivative of Ω at P taken with respect to the directions s_1 and s_2 in the order given.

Thus, if

$$\Omega = 2x^2 - y^2, \qquad l_1 = \frac{2x}{\sqrt{4x^2+1}}, \qquad m_1 = \frac{1}{\sqrt{4x^2+1}},$$

$$l_2 = \frac{y}{\sqrt{x^2+y^2}}, \quad m_2 = \frac{-x}{\sqrt{x^2+y^2}}; \quad D_{s_1}\Omega = \frac{2(4x^2-y)}{\sqrt{4x^2+1}},$$

$$D_{s_2} D_{s_1}\Omega = \frac{x(32x^2y + 8y^2 + 16y + 8x^2 + 2)}{\sqrt{x^2+y^2}\cdot(4x^2+1)^{\frac{3}{2}}}.$$

It is evident from the definition just given that

$$D_{s_2} D_{s_1} \Omega = l_1 \cdot l_2 \cdot \frac{\partial^2 \Omega}{\partial x^2} + (l_1 \cdot m_2 + l_2 \cdot m_1) \frac{\partial^2 \Omega}{\partial x \cdot \partial y} + m_1 \cdot m_2 \cdot \frac{\partial^2 \Omega}{\partial y^2}$$
$$+ \left(l_2 \cdot \frac{\partial l_1}{\partial x} + m_2 \cdot \frac{\partial l_1}{\partial y} \right) \cdot \frac{\partial \Omega}{\partial x} + \left(l_2 \cdot \frac{\partial m_1}{\partial x} + m_2 \cdot \frac{\partial m_1}{\partial y} \right) \frac{\partial \Omega}{\partial y}, \quad (4)$$

and that $D_{s_1} D_{s_2} \Omega$ is quite different in general from $D_{s_2} D_{s_1} \Omega$: the order of the two differentiations is material.

If the *u* curves happen to be a family of parallel straight lines and the *v* curves another family of parallel straight lines,

$$D_{s_2} D_{s_1} \Omega = l_1 l_2 \cdot \frac{\partial^2 \Omega}{\partial x^2} + (l_1 m_2 + l_2 m_1) \cdot \frac{\partial^2 \Omega}{\partial x \cdot \partial y} + m_1 \cdot m_2 \cdot \frac{\partial^2 \Omega}{\partial y^2}, \quad (5)$$

and the coefficients in this expression are constants.

If the *u* curves and the *v* curves are identical and are a family of straight parallel lines, we have

$$D_{s_1}{}^2 \Omega = l_1{}^2 \cdot \frac{\partial^2 \Omega}{\partial x^2} + 2\, l_1 m_1 \cdot \frac{\partial^2 \Omega}{\partial x \cdot \partial y} + m_1{}^2 \cdot \frac{\partial^2 \Omega}{\partial y^2}, \quad (6)$$

the familiar form of the second derivative of Ω along the *fixed direction* s_1, which often appears in work involving the transformation of Cartesian coördinates. Simple special cases of this formula are obtained by putting l equal to 1, 0, and m.

Since $l_1{}^2 + m_1{}^2 = 1$,

$$\frac{\partial l_1}{\partial x} \cdot \frac{\partial m_1}{\partial y} = \frac{\partial m_1}{\partial x} \cdot \frac{\partial l_1}{\partial y},$$

and if at any point s_1 and s_2 are such as to make the coefficient of $\frac{\partial \Omega}{\partial x}$ in (4) vanish, the coefficient of $\frac{\partial \Omega}{\partial y}$ will vanish also. Such points as this lie, in general, on a definite curve, the equation of which is to be found by equating one of these coefficients to zero. If s_1 is a fixed direction so that l_1 and m_1 are constants, (4) takes the form (5), but the coefficients are not constants unless s_2 also is fixed.

If the two variable directions s_1, s_2 coincide, (4) becomes the second derivative of the function Ω taken with respect to the direction s_1; that is,

$$D_{s_1}^2\Omega = l_1^2 \cdot \frac{\partial^2\Omega}{\partial x^2} + 2\,l_1\,m_1 \cdot \frac{\partial^2\Omega}{\partial x \cdot \partial y} + m^2 \cdot \frac{\partial^2\Omega}{\partial y^2}$$
$$+ \left(l_1 \cdot \frac{\partial l_1}{\partial x} + m_1 \cdot \frac{\partial l}{\partial y}\right)\frac{\partial\Omega}{\partial x} + \left(l_1 \cdot \frac{\partial m_1}{\partial x} + m_1 \cdot \frac{\partial m_1}{\partial y}\right)\frac{\partial\Omega}{\partial y}. \quad (7)$$

If the direction cosines of a plane curve at a point on it are l and m, the curvature of the curve at P has the same absolute value as have the expressions

$$\frac{1}{m}\left(l \cdot \frac{\partial l}{\partial x} + m \cdot \frac{\partial l}{\partial y}\right), \quad \frac{1}{l}\left(l \cdot \frac{\partial m}{\partial x} + m \cdot \frac{\partial m}{\partial y}\right). \quad (8)$$

If, therefore, two directions, s_1, s_3, are defined by two curves which, at a point, P, common to both, have a common tangent and equal curvatures, the second derivatives at P of a function Ω taken with respect to the two directions are equal.

If at any point the curvature of the curve of the u family which defines the direction s_1 is zero, the coefficients of $\partial\Omega/\partial x$ and $\partial\Omega/\partial y$ in the expression for $D_{s_1}^2\Omega$ at the point vanish. If the u curves are a family of straight lines, the last two terms of (7) disappear, but the coefficients of the other terms are, in general, not constant.

If there is no point in the region R at which both the quantities $\partial\Omega / \partial x$, $\partial\Omega / \partial y$ vanish together, and if the direction s is at every point of R that in which Ω increases most rapidly, $D_s\Omega = h$, where h is the gradient of Ω, that is, the tensor of the gradient vector. Now h is itself, in general, a scalar point function, which, when equated to a parameter, yields a family of curves the directions of which are usually quite different from those of the lines of the gradient-vector. The normal at any point P to the curve of this h family which passes through the point, has the direction cosines

$$\frac{\partial h}{\partial x}\Big/ h', \quad \frac{\partial h}{\partial y}\Big/ h',$$

where h' is the gradient of h. The angle between the direction, s, of the gradient vector of Ω and the normal to the h curve has at every point the value

$$\cos(\Omega, h) = \left[\frac{\partial h}{\partial x} \cdot \frac{\partial\Omega}{\partial x} + \frac{\partial h}{\partial y} \cdot \frac{\partial\Omega}{\partial y}\right] \Big/ h\,h', \quad (9)$$

and the second derivative of Ω with respect to the direction s is, therefore,

$$D_s^2\Omega = \left[\frac{\partial h}{\partial x} \cdot \frac{\partial\Omega}{\partial x} + \frac{\partial h}{\partial y} \cdot \frac{\partial\Omega}{\partial y}\right] \Big/ h = h' \cdot \cos(\Omega, h). \quad (10)$$

Let the *normal derivative*,* at any point P, of a point function V, taken with respect to another point function W, be the limit, as PQ approaches zero, of the ratio of $V_Q - V_P$ to $W_Q - W_P$, where Q is a point so chosen on the normal at P to the surface of constant W which passes through P, that $W_Q - W_P$ is positive: if, then, (V, W) denotes the angle between the directions in which V and W increase most rapidly, the normal derivatives of V with respect to W, and of W with respect to V, may be written

$$[D_W V] = h_V \cdot \cos(V, W)/h_W, \quad [D_V W] = h_W \cdot \cos(V, W)/h_V: \quad (11)$$

if $h_V = h_W$, these derivatives are equal.

With this notation (10) may be rewritten in the form

$$D_s^2 \Omega = h \cdot [D_\Omega h.] \quad (12)$$

If at any point $D_s^2 \Omega$ vanishes, it is easy to see from (10) that either the gradient (h') of h vanishes at the point, or else the h and Ω surfaces cut each other there orthogonally. This latter case is exemplified in the familiar instance of the electrostatic field due to two long parallel straight wires of the same diameter, charged to equal and opposite potentials: if the wires cut the xy plane normally at P_1, P_2, and if the line joining these intersections be taken for x axis with the point midway between them for origin, the potential function is of the form $V = A \log r_1/r_2$, where $r_1^2 = (x-a)^2 + y^2$, $r_2^2 = (x+a)^2 + y^2$. The intensity of the field, in absolute value, is $h = 2\,a\,A/r_1 r_2$, and the second derivative of V taken along the line of force (that is, the rate at which the intensity of the field changes) is numerically equal to $\dfrac{-4aAx}{r_1^2 \cdot r_2^2}$.

$D_s^2 V$ taken along a line of force vanishes, therefore, at all points on the y axis, and at all such points the curve of constant V ($r_1/r_2 = b$) cuts the curves of constant h ($r_1 r_2 = k$) orthogonally. At points on the y axis the direction of the lines of force is parallel to the x axis, and the second derivative of V with respect to the fixed direction x happens to vanish here also where $l = 1$, $\dfrac{\partial l}{\partial x} = 0$, $m = 0$, $\dfrac{\partial V}{\partial y} = 0$. The quantity h' does not vanish at any finite point.

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* Peirce, The Newtonian Potential Function, p. 116. A Short Table of Integrals, p. 106.

The example just discussed is in contrast with the case where the Ω family are a set of parallel curves of any kind, and h in consequence (if not constant) is a function of Ω alone, so that the h curves and the Ω curves coincide, and if $D_s^2\,\Omega$ vanishes anywhere, it must be where h' vanishes. A simple example of this is furnished by the field of attraction within a very long cylinder of revolution, the density of which is a function of the distance from the axis alone.

If the directions s_1 and s_2 are everywhere perpendicular to each other, we may without loss of generality write $l_2 = -m_1$, $m_2 = l_1$; in which case the coefficients of $\partial\Omega/\partial x$, $\partial\Omega/\partial y$ in (4) become

$$\left(l_2 \cdot \frac{\partial m_2}{\partial x} + m_2 \cdot \frac{\partial m_2}{\partial y}\right) \text{ and } -\left(l_2 \cdot \frac{\partial l_2}{\partial x} + m_2 \cdot \frac{\partial l_2}{\partial y}\right): \tag{13}$$

these vanish if the v curves form a family of straight lines, or the u curves a family of straight or curved parallels. The order of differentiation with respect to the orthogonal directions s_1, s_2 is immaterial if both the u and the v curves are straight lines, that is, if the directions are fixed.

If s_1 is the direction in which Ω increases most rapidly, and s_2 the direction of constant Ω,

$$D_{s_2} D_{s_1}\Omega = D_{s_2} h = \left[\frac{\partial\Omega}{\partial x}\cdot\frac{\partial h}{\partial y} - \frac{\partial\Omega}{\partial y}\cdot\frac{\partial h}{\partial x}\right]\Big/ h$$

$$= \left\{\frac{\partial^2\Omega}{\partial x\cdot\partial y}\left[\left(\frac{\partial\Omega}{\partial x}\right)^2 - \left(\frac{\partial\Omega}{\partial y}\right)^2\right] + \frac{\partial\Omega}{\partial x}\cdot\frac{\partial\Omega}{\partial y}\left[\frac{\partial^2\Omega}{\partial y^2} - \frac{\partial^2\Omega}{\partial x^2}\right]\right\}\Big/ h. \tag{14}$$

Now the direction cosines and the slope of the line of the gradient vector at any point are

$$\frac{1}{h}\cdot\frac{\partial\Omega}{\partial x},\quad \frac{1}{h}\cdot\frac{\partial\Omega}{\partial y},\quad \text{and}\quad \frac{\partial\Omega}{\partial y}\Big/\frac{\partial\Omega}{\partial x}.$$

So that the curvature of the line is

$$\frac{1}{\rho} = \frac{\dfrac{d}{dx}\left[\dfrac{\partial\Omega}{\partial y}\Big/\dfrac{\partial\Omega}{\partial x}\right]}{\left[1 + \left(\dfrac{\partial\Omega}{\partial y}\Big/\dfrac{\partial\Omega}{\partial x}\right)^2\right]^{\frac{3}{2}}}$$

$$= \left\{\frac{\partial^2\Omega}{\partial x\cdot\partial y}\left[\left(\frac{\partial\Omega}{\partial x}\right)^2 - \left(\frac{\partial\Omega}{\partial y}\right)^2\right] + \frac{\partial\Omega}{\partial x}\cdot\frac{\partial\Omega}{\partial y}\left[\frac{\partial^2\Omega}{\partial y^2} - \frac{\partial^2\Omega}{\partial x^2}\right]\right\}\Big/ h^3 \tag{15}$$

and we may write in this case

$$D_{s_2} D_{s_1} \Omega = h / \rho.$$

This expression gives the rate at which the maximum slope of the surface the coördinates of which are (x, y, Ω), changes as one goes along a line of level.*

When s_1 and s_2 are perpendicular to each other, we have in general

$$D_{s_2}{}^2 \Omega = m_1{}^2 \cdot \frac{\partial^2 \Omega}{\partial x^2} - 2 l_1 m_1 \cdot \frac{\partial^2 \Omega}{\partial x \cdot \partial y} + l_1{}^2 \cdot \frac{\partial^2 \Omega}{\partial y^2}$$
$$+ \left(m_1 \cdot \frac{\partial m_1}{\partial x} - l_1 \cdot \frac{\partial m_1}{\partial y} \right) \frac{\partial \Omega}{\partial x} + \left(l_1 \cdot \frac{\partial l_1}{\partial y} - m_1 \cdot \frac{\partial l_1}{\partial x} \right) \frac{\partial \Omega}{\partial y}, \qquad (16)$$

and since $l_1{}^2 + m_1{}^2 = 1$,

$$l_1 \cdot \frac{\partial l_1}{\partial x} + m_1 \cdot \frac{\partial m_1}{\partial x} = 0, \qquad l_1 \cdot \frac{\partial l_1}{\partial y} + m_1 \cdot \frac{\partial m_1}{\partial y} = 0.$$

So that if we add together (7) and (16) we shall get

$$D_{s_1}{}^2 \Omega + D_{s_2}{}^2 \Omega = \frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2} + \frac{1}{m_1} \cdot \frac{\partial l_1}{\partial y} \cdot \frac{\partial \Omega}{\partial x} + \frac{1}{l_1} \cdot \frac{\partial m_1}{\partial x} \cdot \frac{\partial \Omega}{\partial y}$$
$$= \frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2} - \frac{1}{l_1} \cdot \frac{\partial m_1}{\partial y} \cdot \frac{\partial \Omega}{\partial x} - \frac{1}{m_1} \cdot \frac{\partial l_1}{\partial x} \cdot \frac{\partial \Omega}{\partial y}. \qquad (16')$$

It is evident that the values of the space derivatives defined above are wholly independent of the particular system of rectangular coördinates which may be used.

Space Differentiation.

At every point of the space domain, R, let two independent directions (s_1, s_2) be defined by the direction cosines (l_1, m_1, n_1), (l_2, m_2, n_2), where $l_1, m_1, n_1, l_2, m_2, n_2$ are any six single-valued point functions which satisfy the identities

$$l_1{}^2 + m_1{}^2 + n_1{}^2 = 1, \qquad l_2{}^2 + m_2{}^2 + n_2{}^2 = 1, \qquad (17)$$

and have finite derivatives of the first order with respect to the coördinates x, y, z. If, then, Ω is any single-valued function of the coördinates which within R has finite derivatives of the first and second orders with

* Boussinesq, Cours d'Analyse Infinitésimale, T. 1, f. 2, p. 236.

respect to these coördinates, the derivative of Ω at the point P, in the direction s_1, is the value at P of the quantity

$$D_{s_1}\Omega \equiv l_1 \cdot \frac{\partial\Omega}{\partial x} + m_1 \cdot \frac{\partial\Omega}{\partial y} + n_1 \cdot \frac{\partial\Omega}{\partial z}. \tag{18}$$

Through the point P passes a curve of the family defined by the equations

$$\frac{dx}{l_2} = \frac{dy}{m_2} = \frac{dz}{n_2}, \tag{19}$$

and this curve indicates the direction s_2. If on this curve a point Q be taken near P and in the sense of the direction s_2, the limit, as Q approaches P, of the quantity

$$\frac{[D_{s_1}\Omega]_Q - [D_{s_1}\Omega]_P}{\overline{PQ}} \tag{20}$$

may be represented by $[D_{s_2} D_{s_1}\Omega]_P$ and this is the second directional derivative at P of Ω taken with respect to the directions s_1 and s_2 in the order given. It is evident that

$$\begin{aligned} D_{s_2} D_{s_1}\Omega = {} & l_1 l_2 \cdot \frac{\partial^2\Omega}{\partial x^2} + m_1 m_2 \cdot \frac{\partial^2\Omega}{\partial y^2} + n_1 n_2 \cdot \frac{\partial^2\Omega}{\partial z^2} \\ & + (l_1 m_2 + l_2 m_1)\frac{\partial^2\Omega}{\partial x \cdot \partial y} + (m_1 n_2 + m_2 n_1)\frac{\partial^2\Omega}{\partial y \cdot \partial z} + (n_1 l_2 + n_2 l_1)\frac{\partial^2\Omega}{\partial z \cdot \partial x} \\ & + \left(l_2 \cdot \frac{\partial l_1}{\partial x} + m_2 \cdot \frac{\partial l_1}{\partial y} + n_2 \cdot \frac{\partial l_1}{\partial z}\right)\frac{\partial\Omega}{\partial x} \\ & + \left(l_2 \cdot \frac{\partial m_1}{\partial x} + m_2 \cdot \frac{\partial m_1}{\partial y} + n_2 \cdot \frac{\partial m_1}{\partial z}\right)\frac{\partial\Omega}{\partial y} \\ & + \left(l_2 \cdot \frac{\partial n_1}{\partial x} + m_2 \cdot \frac{\partial n_1}{\partial y} + n_2 \cdot \frac{\partial n_1}{\partial z}\right)\frac{\partial\Omega}{\partial z}, \end{aligned} \tag{21}$$

and that this is not equal to $D_{s_1} D_{s_2}\Omega$.

If the directions s_1, s_2 are fixed, the six direction cosines are constants, the last three terms of (21) disappear, and the coefficients of the other six terms are constant. If the fixed directions s_1, s_2 coincide, (21) reduces to the familiar form

$$D_{s_1}^2 \Omega = l_1^2 \cdot \frac{\partial^2 \Omega}{\partial x^2} + m_1^2 \cdot \frac{\partial^2 \Omega}{\partial y^2} + n_1^2 \cdot \frac{\partial^2 \Omega}{\partial z^2}$$

$$+ 2 l_1 m_1 \cdot \frac{\partial^2 \Omega}{\partial x \cdot \partial y} + 2 m_1 n_1 \cdot \frac{\partial^2 \Omega}{\partial y \cdot \partial z} + 2 l_1 n_1 \cdot \frac{\partial^2 \Omega}{\partial x \cdot \partial z}, \quad (22)$$

whereas, if s_1 is not fixed,

$$D_{s_1}^2 \Omega = l_1^2 \cdot \frac{\partial^2 \Omega}{\partial x^2} + m_1^2 \cdot \frac{\partial^2 \Omega}{\partial y^2} + n_1^2 \cdot \frac{\partial^2 \Omega}{\partial z^2} + 2 l_1 m_1 \cdot \frac{\partial^2 \Omega}{\partial x \cdot \partial y} + 2 m_1 n_1 \cdot \frac{\partial^2 \Omega}{\partial y \cdot \partial z}$$

$$+ 2 l_1 n_1 \cdot \frac{\partial^2 \Omega}{\partial x \cdot \partial z} + \left(l_1 \cdot \frac{\partial l_1}{\partial x} + m_1 \cdot \frac{\partial l_1}{\partial y} + n_1 \cdot \frac{\partial l_1}{\partial z} \right) \frac{\partial \Omega}{\partial x}$$

$$+ \left(l_1 \cdot \frac{\partial m_1}{\partial x} + m_1 \cdot \frac{\partial m_1}{\partial y} + n_1 \cdot \frac{\partial m_1}{\partial z} \right) \frac{\partial \Omega}{\partial y}$$

$$+ \left(l_1 \cdot \frac{\partial n_1}{\partial x} + m_1 \cdot \frac{\partial n_1}{\partial y} + n_1 \cdot \frac{\partial n_1}{\partial z} \right) \frac{\partial \Omega}{\partial z}. \quad (23)$$

All the coefficients in (22) are constants; all those of (23) are in general variable. If s_1 is defined by any infinite system of straight lines of which just one passes through every point of space, and if the direction s_1 at all points of any one of the lines is that of the line itself, the coefficients of $\partial \Omega / \partial x$, $\partial \Omega / \partial y$, $\partial \Omega / \partial z$ in (23) vanish. In particular, if the direction s_1 is that of the radius vector from a fixed point (a, b, c), (23) takes the form of (22) though the remaining coefficients are not constants. In any case if the coefficients of two of the three quantities $\partial \Omega / \partial x$, $\partial \Omega / \partial y$, $\partial \Omega / \partial z$ vanish, the third must vanish also.

If the gradient, h, of Ω does not vanish at any point of R and if s is the direction in which Ω increases most rapidly,

$$D_s \Omega = h,$$

$$D_s^2 \Omega = \left[\frac{\partial h}{\partial x} \cdot \frac{\partial \Omega}{\partial x} + \frac{\partial h}{\partial y} \cdot \frac{\partial \Omega}{\partial y} + \frac{\partial h}{\partial z} \cdot \frac{\partial \Omega}{\partial z} \right] \Big/ h. \quad (24)$$

If h' is the gradient of the scalar point function which gives the value of h, and if (Ω, h) represents the angle between the directions in which the point functions Ω and h increase most rapidly,

$$\cos(\Omega, h) = \left[\frac{\partial h}{\partial x} \cdot \frac{\partial \Omega}{\partial x} + \frac{\partial h}{\partial y} \cdot \frac{\partial \Omega}{\partial y} + \frac{\partial h}{\partial z} \cdot \frac{\partial \Omega}{\partial z} \right] \Big/ h \cdot h' \quad (25)$$

and $$D_s^2 \Omega = h' \cdot \cos(\Omega, h), \text{ or } h\,[D_\Omega h] \quad (26)$$

where $[D_\Omega h]$ represents the normal derivative of h with respect to Ω.

If the equation $\Omega = k$ happens to represent a set of parallel surfaces, h, if not constant, is a function of Ω alone, so that the h and Ω surfaces are coincident: in this case $\cos(\Omega, h) = 1$ and $D_s^2\,\Omega$ can vanish only where h' vanishes. In general, $D_s^2\,\Omega$ vanishes when the h and Ω surfaces cut each other at right angles.

If s_1, s_2, s_3 are any three mutually perpendicular directions,

$$l_1^2 + l_2^2 + l_3^2 = m_1^2 + m_2^2 + m_3^2 = n_1^2 + n_2^2 + n_3^2 = 1,$$

$$l_1 m_1 + l_2 m_2 + l_3 m_3 = m_1 n_1 + m_2 n_2 + m_3 n_3 = l_1 n_1 + l_2 n_2 + l_3 n_3 = 0,$$

and

$$D_{s_1}^2\Omega + D_{s_2}^2\Omega + D_{s_3}^2\Omega = \frac{\partial^2\Omega}{\partial x^2} + \frac{\partial^2\Omega}{\partial y^2} + \frac{\partial^2\Omega}{\partial z^2}$$

$$+ \frac{\partial\Omega}{\partial x}\left[m_1 \cdot \frac{\partial l_1}{\partial y} + m_2 \cdot \frac{\partial l_2}{\partial y} + m_3 \cdot \frac{\partial l_3}{\partial y} + n_1 \cdot \frac{\partial l_1}{\partial z} + n_2 \cdot \frac{\partial l_2}{\partial z} + n_3 \cdot \frac{\partial l_3}{\partial z}\right]$$

$$+ \frac{\partial\Omega}{\partial y}\left[n_1 \cdot \frac{\partial m_1}{\partial z} + n_2 \cdot \frac{\partial m_2}{\partial z} + n_3 \cdot \frac{\partial m_3}{\partial z} + l_1 \cdot \frac{\partial m_1}{\partial x} + l_2 \cdot \frac{\partial m_2}{\partial x} + l_3 \cdot \frac{\partial m_3}{\partial x}\right]$$

$$+ \frac{\partial\Omega}{\partial z}\left[l_1 \cdot \frac{\partial n_1}{\partial x} + l_2 \cdot \frac{\partial n_2}{\partial x} + l_3 \cdot \frac{\partial n_3}{\partial x} + m_1 \cdot \frac{\partial n_1}{\partial y} + m_2 \cdot \frac{\partial n_2}{\partial y} + m_3 \cdot \frac{\partial n_3}{\partial y}\right]. \tag{27}$$

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